



ELSEVIER

Linear Algebra and its Applications 359 (2003) 169–179

**LINEAR ALGEBRA
AND ITS
APPLICATIONS**

www.elsevier.com/locate/laa

On the operator equation $e^A = e^B$

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Received 14 April 2000; accepted 30 April 2002

Submitted by R.A. Brualdi

Abstract

Suppose that A and B are bounded linear operators on a complex Hilbert space and that $e^A = e^B$. It is well-known that if the spectrum of A is incongruent (mod $2\pi i$) then $AB = BA$. In this note we show that if A is normal and $\|A\| \leq \pi$ then $e^A = e^B$ implies that $A^2B = BA^2$. If B is also normal, $\|B\| \leq \pi$ and $-\pi i$ is not an eigenvalue of A then we show that $e^A = e^B$ implies $AB = BA$ and $(A - B)^2 = 2\pi i(A - B)$.

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AMS classification: 47A10; 47A60

Keywords: Normal operators; Exponentials

Throughout this paper let \mathcal{H} denote a complex Hilbert space and $\mathcal{L}(\mathcal{H})$ the Banach algebra of all bounded linear operators on \mathcal{H} . For $A \in \mathcal{L}(\mathcal{H})$ the spectrum, the set of eigenvalues, and the spectral radius of A are denoted by $\sigma(A)$, $\sigma_p(A)$, and $r(A)$, respectively. For the resolvent set of A we write $\rho(A)$. We use $N(A)$ and $A(\mathcal{H})$ to denote the kernel and the range of A , respectively.

We say that $\sigma(A)$ is *incongruent* (mod $2\pi i$), if

$$\sigma(A) \cap \sigma(A + 2j\pi i) = \emptyset \quad \text{for } j = \pm 1, \pm 2, \dots$$

If $A \in \mathcal{L}(\mathcal{H})$ is *normal* ($AA^* = A^*A$) and has the spectral resolution

$$A = \int_{\sigma(A)} \lambda dE(\lambda), \tag{1}$$

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let $E(\Omega)$ denote the associated projection measure defined on the Borel subsets $\Omega \subseteq \sigma(A)$. It is convenient to think of E as being defined for all Borel subsets in \mathbb{C} : put $E(\Omega) = E(\Omega \cap \sigma(A))$. It is well-known that $\|A\| = r(A)$ if A is normal.

Let $T \in \mathcal{L}(\mathcal{H})$. The map $\delta_T : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$, defined by

$$\delta_T(C) = CT - TC \quad (C \in \mathcal{L}(\mathcal{H}))$$

is called the *inner derivation* determined by T . δ_T is a bounded linear operator on $\mathcal{L}(\mathcal{H})$ with $\|\delta_T\| \leq 2\|T\|$.

It is shown in [3] that

$$\sigma(\delta_T) = \{\lambda - \mu : \lambda, \mu \in \sigma(T)\}, \quad (2)$$

where $\sigma(\delta_T)$ denotes the spectrum of δ_T . From [4, Proposition 6.4.8] it follows that

$$e^{\delta_T}(C) = e^{-T} C e^T \quad \text{for all } C \in \mathcal{L}(\mathcal{H}). \quad (3)$$

Throughout this note let f denote the entire function $f : \mathbb{C} \rightarrow \mathbb{C}$ given by

$$f(z) = \begin{cases} z^{-1}(e^z - 1), & \text{if } z \neq 0, \\ 1, & \text{if } z = 0. \end{cases}$$

From $zf(z) = f(z)z = e^z - 1$ and (3) we get

$$f(\delta_T)(\delta_T(C)) = e^{-T} C e^T - C \quad \text{for all } C \in \mathcal{L}(\mathcal{H}). \quad (4)$$

For a bounded linear operator \mathcal{T} on $\mathcal{L}(\mathcal{H})$ we denote the kernel of \mathcal{T} by $N(\mathcal{T})$.

Proposition 1. *Let $A \in \mathcal{L}(\mathcal{H})$ such that $r(A) \leq \pi$. Let*

$$M_A = \{\lambda \in \sigma(\delta_A) : f(\lambda) = 0\}.$$

- (a) *If $\sigma(A)$ is incongruent (mod $2\pi i$), then $M_A = \emptyset$.*
- (b) *If $M_A = \emptyset$, then $f(\delta_A)$ is an invertible operator on $\mathcal{L}(\mathcal{H})$.*
- (c) $M_A \subseteq \{2\pi i, -2\pi i\}$.
- (d) $N(f(\delta_A)) = N(\delta_A - 2\pi i) \oplus N(\delta_A + 2\pi i)$.
- (e) *If $C \in N(\delta_A - 2\pi i)$ or $C \in N(\delta_A + 2\pi i)$, then $C^2 = 0$.*

Proof. (a) and (b) are clear.

(c) Take $\lambda \in M_A$, then $0 \neq \lambda \in \sigma(\delta_A)$ and $e^\lambda = 1$. Thus $\lambda = 2j\pi i$ for some $j \in \mathbb{Z} \setminus \{0\}$. It follows from (2) that there are $\alpha, \beta \in \sigma(A)$ such that $\lambda = \alpha - \beta$. Hence $2|j|\pi = |\lambda| \leq |\alpha| + |\beta| \leq 2r(A) \leq 2\pi$, therefore $|j| \leq 1$. This shows that $\lambda \in \{2\pi i, -2\pi i\}$.

(d) Since $2\pi i$ and $-2\pi i$ are simple zeros of f , there is an entire function g such that

$$f(\lambda) = g(\lambda)(\lambda - 2\pi i)(\lambda + 2\pi i)$$

and $g(\lambda) \neq 0$ for all $\lambda \in \sigma(\delta_A)$, thus $g(\delta_A)$ is invertible on $\mathcal{L}(\mathcal{H})$ and

$$f(\delta_A) = g(\delta_A)(\delta_A - 2\pi i)(\delta_A + 2\pi i).$$

Satz 80.3 in [1] gives now

$$N(f(\delta_A)) = N(\delta_A - 2\pi i) \oplus N(\delta_A + 2\pi i).$$

(e) Take $C \in N(\delta_A - 2\pi i)$, thus $CA - AC = 2\pi iC$. Therefore $2\pi iC^2 = C^2A - CAC = C^2A - (AC + 2\pi iC)C = C^2A - AC^2 - 2\pi iC^2$, hence

$$C^2A - AC^2 = 4\pi iC^2.$$

Assume that $C^2 \neq 0$, then $4\pi i \in \sigma(\delta_A)$, therefore, by (2), $4\pi \leq 2r(A) \leq 2\pi$, a contradiction. Thus $C^2 = 0$. \square

Notation. Let $T \in \mathcal{L}(\mathcal{H})$, $\Omega \subseteq \mathbb{C}$ and $\Omega \neq \emptyset$. Let $S(T, \Omega)$ be the subset of \mathcal{H} defined by

$$S(T, \Omega) = \bigcap_{\lambda \in \Omega} (T - \lambda)(\mathcal{H}).$$

Proposition 2. Let $A \in \mathcal{L}(\mathcal{H})$ be normal.

- (a) For $\mu \in \mathbb{C}$, $(A - \mu)(\mathcal{H}) = (A^* - \bar{\mu})(\mathcal{H})$.
- (b) If A has the spectral resolution (1) and if $B \in \mathcal{L}(\mathcal{H})$, then

$$S(A, \rho(B)) = E(\sigma(A) \cap \sigma(B))(\mathcal{H}).$$

Proof. (a) Since A is normal, $A - \mu$ is normal. Use Exercise 12.36 in [6] to see that $(A - \mu)(\mathcal{H}) = (A^* - \bar{\mu})(\mathcal{H})$.

(b) follows from Theorem 1 in [5]. \square

Proposition 3. Let $A \in \mathcal{L}(\mathcal{H})$ be normal and suppose that A has the spectral resolution (1). For $\lambda_0 \in \mathbb{C}$, $C \in N(\delta_A - \lambda_0)$ and $D \in N(\delta_A + \lambda_0)$ we have

$$C(\mathcal{H}) \subseteq E(\sigma(A) \cap \sigma(A - \lambda_0))(\mathcal{H})$$

and

$$D^*(\mathcal{H}) \subseteq E(\sigma(A) \cap \sigma(A - \lambda_0))(\mathcal{H}).$$

Proof. Take $C \in \mathcal{L}(\mathcal{H})$ with $CA - AC = \lambda_0 C$, thus $AC = C(A - \lambda_0)$. Put $B = A - \lambda_0$. For $\mu \in \rho(B)$ we get

$$\begin{aligned} (A - \mu)C(B - \mu)^{-1} &= AC(B - \mu)^{-1} - \mu C(B - \mu)^{-1} \\ &= CB(B - \mu)^{-1} - \mu C(B - \mu)^{-1} \\ &= C(B - \mu)(B - \mu)^{-1} = C. \end{aligned}$$

This shows that $C(\mathcal{H}) \subseteq S(A, \rho(B))$. From Proposition 2(b) we get

$$S(A, \rho(B)) = E(\sigma(A) \cap \sigma(B))(\mathcal{H}), \tag{5}$$

thus $C(\mathcal{H}) \subseteq E(\sigma(A) \cap \sigma(A - \lambda_0))(\mathcal{H})$.

Now take $D \in N(\delta_A + \lambda_0)$, hence $DA - AD = -\lambda_0 D$ thus $DA = (A - \lambda_0)D$. As above let $B = A - \lambda_0$. Then $A^*D^* = D^*B^*$. A similar computation as above gives

$$(A^* - \mu)D^*(B^* - \mu)^{-1} = D^* \text{ for all } \mu \in \rho(B^*).$$

Thus $D^*(\mathcal{H}) \subseteq S(A^*, \rho(B^*))$. Since $\rho(B^*) = \{\mu \in \mathbb{C} : \bar{\mu} \in \rho(B)\}$, we get from Proposition 2(a) that

$$S(A^*, \rho(B^*)) = S(A, \rho(B)),$$

hence, by (5), $D^*(\mathcal{H}) \subseteq E(\sigma(A) \cap \sigma(A - \lambda_0))(\mathcal{H})$. \square

Proposition 4. Let $A \in \mathcal{L}(\mathcal{H})$ be normal and $r(A) \leq \pi$.

(a) If $C \in N(\delta_A + 2\pi i)$ then $AC = i\pi C$.

(b) If $D \in N(\delta_A - 2\pi i)$ then $AD^* = i\pi D^*$ and $DA = i\pi D$.

Proof. Suppose that A has the spectral resolution (1). It is clear that

$$\sigma(A) \cap \sigma(A + 2\pi i) \subseteq \{i\pi\}.$$

From Theorem 12.29 in [6] it follows that $E(\{i\pi\}) = N(A - i\pi)$. Thus

$$E(\sigma(A) \cap \sigma(A + 2\pi i))(\mathcal{H}) \subseteq N(A - i\pi). \quad (6)$$

(a) Take $C \in N(\delta_A + 2\pi i)$ and put $\lambda_0 = -2\pi i$. Then $C \in N(\delta_A - \lambda_0)$. Proposition 3 and (6) give

$$C(\mathcal{H}) \subseteq N(A - i\pi),$$

hence $AC = i\pi C$.

(b) Take $D \in N(\delta_A - 2\pi i)$ and put $\lambda_0 = -2\pi i$. Then $D \in N(\delta_A + \lambda_0)$. Proposition 3 and (6) give

$$D^*(\mathcal{H}) \subseteq N(A - i\pi),$$

hence $AD^* = i\pi D^*$. Therefore we have $AD^*x = i\pi D^*x$ for each $x \in \mathcal{H}$. The normality of A gives $A^*D^*x = -i\pi D^*x$ for all $x \in \mathcal{H}$, thus $A^*D^* = -i\pi D^*$, hence $DA = i\pi D$. \square

We now are in a position to state the main results of this paper.

The following theorem is due to Hille [2]. For the convenience of the reader we shall include a proof.

Theorem 1. Let $A, B \in \mathcal{L}(\mathcal{H})$ and $e^A = e^B$. If $\sigma(A)$ is incongruent (mod $2\pi i$) then $AB = BA$.

Proof. From (4) we get

$$f(\delta_A)(\delta_A(B)) = e^{-A}Be^A - B = 0,$$

thus $AB - BA \in N(f(\delta_A))$. Use Proposition 1(a) and (b) to see that $AB = BA$. \square

The restriction concerning the spectrum of A in Theorem 1 cannot be dispensed with, as is seen by the following two-dimensional example.

Example. Let

$$A = \pi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \pi \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}.$$

By induction we see that

$$A^{2n} = (-1)^n \pi^{2n} I = B^{2n}, \quad A^{2n+1} = (-1)^n \pi^{2n} A,$$

and

$$B^{2n+1} = (-1)^n \pi^{2n} B \quad \text{for } n = 0, 1, 2, \dots$$

This shows that $e^A = -I = e^B$. We have

$$AB = \pi^2 \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \neq \pi^2 \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = BA.$$

It is easily seen that A is normal and $\sigma(A) = \{i\pi, -i\pi\}$, thus $r(A) = \pi$ and $\sigma(A)$ is not incongruent (mod $2\pi i$). But we have $A^2 B = B A^2$, which will be the case in general as the following theorem shows.

Theorem 2. Suppose that $A \in \mathcal{L}(\mathcal{H})$ is normal, $B \in \mathcal{L}(\mathcal{H})$ and $e^A = e^B$.

- (a) If $r(A) < \pi$ then $AB = BA$.
- (b) If $r(A) = \pi$ then $A^2 B = B A^2$.
- (c) If $r(A) \leq \pi$ and $i\pi \notin \sigma_p(A)$ then $AB = BA$.
- (d) If $r(A) \leq \pi$ and $-i\pi \notin \sigma_p(A)$ then $AB = BA$.

Proof. As in the proof of Theorem 1 we get $AB - BA \in N(f(\delta_A))$.

(a) If $r(A) < \pi$, then $\sigma(A)$ is incongruent (mod $2\pi i$), hence $AB = BA$, by Theorem 1.

(b) From Proposition 1 we see that there are operators $C \in N(\delta_A + 2\pi i)$ and $D \in N(\delta_A - 2\pi i)$ such that $AB - BA = C + D$. From $CA - AC = -2\pi i C$ and Proposition 4(a) we derive $AC = i\pi C$, hence

$$CA = AC - 2\pi i C = -i\pi C = -AC,$$

thus

$$AC + CA = 0. \tag{7}$$

Use Proposition 4(b) and $DA - AD = 2\pi i D$ to get $DA = i\pi D$ and

$$AD = DA - 2\pi i D = -i\pi D = -DA,$$

hence

$$AD + DA = 0. \quad (8)$$

From $AB - BA = C + D$ it follows that

$$A^2B - ABA = AC + AD \quad (9)$$

and

$$ABA - BA^2 = CA + DA. \quad (10)$$

The addition of (9) and (10) shows

$$A^2B - BA^2 = AC + CA + AD + DA.$$

Now use (7) and (8) to get $A^2B = BA^2$.

(c) As in (b) we have $AB - BA = C + D$ with $C \in N(\delta_A + 2\pi i)$ and $D \in N(\delta_A - 2\pi i)$. Since $N(A - i\pi) = \{0\}$ we conclude from Proposition 4 that $C = 0 = D^*$, thus $C = D = 0$, hence $AB = BA$.

(d) If $-i\pi \notin \sigma_p(A)$ then $i\pi \notin \sigma_p(-A)$. From $e^{-A} = e^{-B}$ we get then $AB = BA$ as in the proof of (c). \square

Now suppose that the spectrum $\sigma(A)$ of $A \in \mathcal{L}(\mathcal{H})$ satisfies

$$\sigma(A) \subseteq \{z \in \mathbb{C} : |\operatorname{Im} z| \leq \pi\} \quad (11)$$

and

$$\sigma(A) \cap \sigma(A + 2\pi i) \subseteq \{i\pi\}. \quad (12)$$

Then it is easily seen that $M_A \subseteq \{2\pi i, -2\pi i\}$ and

$$N(f(\delta_A)) = N(\delta_A - 2\pi i) \oplus N(\delta_A + 2\pi i).$$

Let A be normal and suppose that (11) and (12) hold. Then it is easy to see that the statements of Proposition 4 remain valid. Thus an inspection of the proof of Theorem 2 shows the following result:

Theorem 3. Suppose that $A \in \mathcal{L}(\mathcal{H})$ is normal, $B \in \mathcal{L}(\mathcal{H})$, $e^A = e^B$ and $\sigma(A)$ satisfies (11) and (12). Then $A^2B = BA^2$. If $i\pi \notin \sigma_p(A)$ or $-i\pi \notin \sigma_p(A)$ then $AB = BA$.

Theorem 4. Let $A, B \in \mathcal{L}(\mathcal{H})$, let $e^A = e^B$ and let $\sigma(A)$ and $\sigma(A - B)$ be incongruent (mod $2\pi i$). Then there is some $k \in \mathbb{Z}$ with

$$A - B = (2k\pi i)I.$$

Proof. From Theorem 1 we get $AB = BA$, thus $e^{A-B} = I$. Put $C = A - B$ and let $g(z) = e^z - 1$ ($z \in \mathbb{C}$). Hence $g(C) = 0$. Take $\lambda, \mu \in \sigma(C)$, then $e^\lambda = e^\mu = 1$, thus $\lambda - \mu = 2j\pi i$ for some $j \in \mathbb{Z}$. Since $\sigma(C)$ is incongruent (mod $2\pi i$),

we get $\lambda = \mu$. This shows that there is $k \in \mathbb{Z}$ such that $\sigma(C) = \{2k\pi i\}$. Since $2k\pi i$ is a simple zero of g , there is an entire function h with $g(\lambda) = h(\lambda)(\lambda - 2k\pi i)$ and $h(2k\pi i) \neq 0$.

This gives

$$0 = g(C) = h(C)(C - 2k\pi i).$$

Since $h(C)$ is invertible, $C = 2k\pi iI$. \square

As an immediate consequence of Theorem 4 we have the following well-known result:

Corollary 1. *If $A, B \in \mathcal{L}(\mathcal{H})$ are selfadjoint and if $e^A = e^B$ then $A = B$.*

Proof. From $\sigma(A), \sigma(A - B) \subseteq \mathbb{R}$ we see that $\sigma(A)$ and $\sigma(A - B)$ are incongruent (mod $2\pi i$). Theorem 4 gives $A - B = 2k\pi iI$ for some $k \in \mathbb{Z}$. Since $A - B = (A - B)^* = -2k\pi iI = B - A$, $A = B$. \square

Corollary 2. *If $A, B \in \mathcal{L}(\mathcal{H})$ are normal and if $e^A = e^B$ then $A + A^* = B + B^*$.*

Proof. Since A and B are normal, we see that

$$\begin{aligned} e^{A+A^*} &= e^A e^{A^*} = e^A (e^A)^* = e^B (e^B)^* \\ &= e^B e^{B^*} = e^{B+B^*}. \end{aligned}$$

We now use Corollary 1. \square

For our next result we need the following proposition.

Proposition 5. *Suppose that A and B are normal operators in $\mathcal{L}(\mathcal{H})$, $r(A) \leq \pi$, $r(B) \leq \pi$ and $AB = BA$. Then*

- (a) $A - B$ is normal;
- (b) $N(A - B - 2\pi i) = N(A - i\pi) \cap N(B + i\pi)$.

Proof. (a) From $AB = BA$ we get $AB^* = B^*A$ and $A^*B = BA^*$ by the Fuglede–Putnam–Rosenblum Theorem [6, Theorem 12.16]. A simple computation gives then that $(A - B)(A - B)^* = (A - B)^*(A - B)$.

(b) It is clear that $N(A - i\pi) \cap N(B + i\pi) \subseteq N(A - B - 2\pi i)$. Put $C = A - B$ and take $x_0 \in N(A - B - 2\pi i)$. We can assume that $\|x_0\| = 1$. For the following computations let $(\cdot | \cdot)$ denote the inner product \mathcal{H} . From $Cx_0 = 2\pi i x_0$ and (a) we get $C^*x_0 = -2\pi i x_0$, thus

$$Ax_0 = Bx_0 + 2\pi i x_0 \quad \text{and} \quad A^*x_0 = B^*x_0 - 2\pi i x_0. \quad (13)$$

Put $D = i(B - B^*)$. Then $D^* = D$ and

$$\begin{aligned}(Dx_0|x_0) &= i((Bx_0|x_0) - \overline{(Bx_0|x_0)}) \\ &= -2 \operatorname{Im}(Bx_0|x_0).\end{aligned}$$

From $|\operatorname{Im}(Bx_0|x_0)| \leq |(Bx_0|x_0)| \leq \|B\| = r(B) \leq \pi$, we see that $-\pi \leq \operatorname{Im}(Bx_0|x_0)$. Thus

$$(Dx_0|x_0) \leq 2\pi. \quad (14)$$

Now use (13) and (14) to derive

$$\begin{aligned}\|(A - i\pi)x_0\|^2 &= ((A^* + i\pi)(A - i\pi)x_0|x_0) \\ &= \|Ax_0\|^2 + i\pi((A - A^*)x_0|x_0) + \pi^2 \\ &\leq 2\pi^2 + i\pi((A - A^*)x_0|x_0) \\ &= 2\pi^2 + i\pi((B - B^* + 4\pi i)x_0|x_0) \\ &= -2\pi^2 + \pi(Dx_0|x_0) \\ &\leq -2\pi^2 + 2\pi^2 = 0.\end{aligned}$$

Thus $x_0 \in N(A - i\pi)$, hence, by (13), $x_0 \in N(B + i\pi)$. \square

Theorem 5. Suppose that A and B are normal operators in $\mathcal{L}(\mathcal{H})$, $r(A) \leq \pi$, $r(B) \leq \pi$ and $e^A = e^B$.

- (a) If $i\pi \notin \sigma_p(A)$ then $AB = BA$ and $-(1/2\pi i)(A - B)$ is an orthogonal projection.
- (b) If $-i\pi \notin \sigma_p(A)$ then $AB = BA$ and $(1/2\pi i)(A - B)$ is an orthogonal projection.
- (c) If $-i\pi \notin \sigma_p(A)$ and $i\pi \notin \sigma_p(A)$ then $A = B$.
- (d) If $-i\pi \notin \sigma_p(A)$ and $-i\pi \notin \sigma_p(B)$ then $A = B$.
- (e) If $i\pi \notin \sigma_p(A)$ and $i\pi \notin \sigma_p(B)$ then $A = B$.

Proof. Let $C = A - B$.

(a) Theorem 2 gives $AB = BA$, hence $e^C = I$. Use Proposition 5 to see that C is normal. Define the polynomial p by $p(z) = z(z + 2\pi i)(z - 2\pi i)$. Then $p(C)$ is normal and therefore

$$\|p(C)\| = r(p(C)). \quad (15)$$

Now take $\lambda \in \sigma(C)$. Since $e^C = I$, $\lambda = 2j\pi i$ for some $j \in \mathbb{Z}$. Thus

$$2|j|\pi = |\lambda| \leq \|C\| \leq \|A\| + \|B\| = r(A) + r(B) \leq 2\pi,$$

hence $j \in \{0, 1, -1\}$, thus $\sigma(C) \subseteq \{0, 2\pi i, -2\pi i\}$. The spectral mapping theorem gives now

$$\sigma(p(C)) = p(\sigma(C)) = \{0\},$$

hence, by (15),

$$(C - 2\pi i)(C + 2\pi i)C = 0. \quad (16)$$

Since $N(A - i\pi) = \{0\}$, we see from Proposition 5(b) and (16) that $C^2 = -2\pi iC$. Put $P = -(1/2\pi i)C$. Then $P^2 = P$, thus P is a projection. It remains to show that $P = P^*$. But this follows from Corollary 2, since

$$P^* = \frac{1}{2\pi i}(A^* - B^*) = -\frac{1}{2\pi i}(A - B) = P.$$

(b) If $-i\pi \notin \sigma_p(A)$ then $i\pi \notin \sigma_p(-A)$. Since $e^{-A} = e^{-B}$, (a) shows that $-(1/2\pi i)(-A - (-B)) = (1/2\pi i)(A - B)$ is an orthogonal projection.

(c) From (a) and (b) we conclude that $-2\pi iC = C^2 = 2\pi iC$, hence $C = 0$.

(d) From (b) we get $C^2 = 2\pi iC$. Use (b) with B instead of A to derive $C^2 = (B - A)^2 = (1/2\pi i)(B - A) = -2\pi iC$. Hence $C = 0$.

(e) is now clear.

Remark. Theorem 5 generalizes Satz 4 in [7].

For our final result in this paper we return to the situation of Theorem 3.

Theorem 6. Suppose that $A \in \mathcal{L}(\mathcal{H})$ is normal, $B \in \mathcal{L}(\mathcal{H})$, $e^A = e^B$ and $\sigma(A)$ satisfies (11) and (12). Then we have:

- (a) $A^{2n}B = BA^{2n}$ for all $n \in \mathbb{N}$.
- (b) $A^{2n+1}B - BA^{2n+1} = A^{2n}(AB - BA) = (-1)^n \pi^{2n}(AB - BA)$ for all $n \in \mathbb{N}$.
- (c) $e^{2A}(AB - BA) = e^{2B}(AB - BA) = AB - BA$.
- (d) If $AB \neq BA$ then there is some $k \in \mathbb{Z}$ such that $k\pi i \in \sigma_p(B)$.

Proof. (a) Follows from Theorem 3.

(b) As in the proof of Theorem 2 there are operators $C \in N(\delta_A + 2\pi i)$ and $D \in N(\delta_A - 2\pi i)$ such that $AB - BA = C + D$, $AC = i\pi C$ and $DA = i\pi D$. It follows that

$$\begin{aligned} i\pi(AB - BA) &= i\pi C + i\pi D = AC + DA \\ &= AC + AD + 2\pi iD \\ &= A(AB - BA) + 2\pi iD, \end{aligned}$$

thus

$$(A - i\pi)(AB - BA) = -2\pi iD.$$

Since $D(A - i\pi) = 0$,

$$(A - i\pi)(AB - BA)(A - i\pi) = 0.$$

This and

$$\begin{aligned} A(AB - BA) &= A^2B - ABA = BA^2 - ABA \\ &= (BA - AB)A = -(AB - BA)A \end{aligned}$$

show that

$$0 = (A - i\pi)((-A - i\pi)(AB - BA)) = -(A^2 + \pi^2)(AB - BA),$$

hence

$$A^2(AB - BA) = -\pi^2(AB - BA). \quad (17)$$

Take $n \in \mathbb{N}$. Then, by (a) and (17),

$$\begin{aligned} A^{2n+1}B - BA^{2n+1} &= A^{2n+1}B - A^{2n}BA = A^{2n}(AB - BA) \\ &= (-1)^n \pi^{2n}(AB - BA). \end{aligned}$$

(c) Since (b) holds we get

$$\begin{aligned} \frac{1}{2}(e^A - e^{-A})(AB - BA) &= \sinh(A)(AB - BA) \\ &= \sum_{n=0}^{\infty} \frac{A^{2n+1}(AB - BA)}{(2n+1)!} \\ &= A \sum_{n=0}^{\infty} \frac{A^{2n}(AB - BA)}{(2n+1)!} \\ &= A \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}(AB - BA)}{(2n+1)!} \\ &= \frac{A}{\pi} \left(\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} \right) (AB - BA) \\ &= \frac{\sin \pi}{\pi} A(AB - BA) = 0. \end{aligned}$$

Thus $e^A(AB - BA) = e^{-A}(AB - BA)$. From $e^A = e^B$ we then derive

$$e^{2B}(AB - BA) = e^{2A}(AB - BA) = AB - BA.$$

(d) Suppose that $AB \neq BA$. From (c) we see that $1 \in \sigma_p(e^{2B})$. The spectral mapping theorem for the point spectrum ([6, Theorem 10.33]) shows that there is $\lambda \in \sigma_p(B)$ such that $e^{2\lambda} = 1$, hence $2\lambda = 2k\pi i$ for some $k \in \mathbb{Z}$. This gives $k\pi i \in \sigma_p(B)$. \square

Corollary 3. Suppose that $A \in \mathcal{L}(\mathcal{H})$ is normal, $B \in \mathcal{L}(\mathcal{H})$, $e^A = e^B$, $\sigma(A)$ satisfies (11) and (12) and $\sigma(B)$ satisfies

$$\sigma_p(B) \cap \{k\pi i : k \in \mathbb{Z}\} = \emptyset$$

then $AB = BA$.

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